Maximum Turning Angle across an Oblique Shock

Philip A. Thompson*
Rensselaer Polytechnic Institute, Troy, N. Y.

The geometry of an oblique shock is fixed by the density ratio across the shock, together with the invariance of the tangential component of velocity. It follows that, for a shock of fixed density ratio, the maximum turning angle θ_m and corresponding shock angle β_m are uniquely determined without placing any restrictions on the equation of state. The resulting algebraic expressions are simple enough to be useful.

The geometry of the oblique shock is shown in Fig. 1. By inspection, $w_1 = v\tan\beta$ and $w_2 = v\tan(\beta - \theta)$. Dividing the second relation into the first, and making use of $w_1/w_2 = \rho_2/\rho_1$ (by conservation of mass across the shock) yields

$$r = \tan\beta/\tan(\beta - \theta) \quad (1)$$

where $r \equiv \rho_2/\rho_1$ is the density ratio (for compression shocks, $r \geq 1$). Equation (1) appears in standard books on gasdynamics.[1,2] Trigonometric manipulation yields an explicit formula for $\theta(\beta,r)$,

$$\tan\theta = (r - 1)\tan\beta/(r + \tan^2\beta) \quad (2)$$

Consider now changing the shock angle β while holding r constant (this corresponds to keeping the shock strength fixed, or considering the same physical shock in different reference frames). From elementary calculus, the turning angle θ is an extremum when $d\theta/d\beta = 0$. Differentiating Eq. (2) with respect to β and setting $d\theta/d\beta = 0$ then yields

$$\tan\beta_m = r^{1/2} \quad (3)$$

which is the shock angle corresponding to maximum turning. The associated maximum turning angle θ_m is then found by substitution into Eq. (3),

$$\tan\theta_m = (r - 1)/2r^{1/2} \quad (4)$$

As a numerical example, consider a shock with $P_1 = 1$ atm and $P_2 = 50{,}000$ atm. In a perfect gas with $\gamma = 1.40$ the density ratio is the limiting value $r = 6$, yielding $\beta_m = 67.79°$, $\theta_m = 45.58°$. In water the density ratio (assuming an initial state near room temperature) is found to be $r \approx 1.51$, yielding $\beta_m = 50.8°$, $\theta_m = 11.7°$.

For certain problems, involving streaming flows in particular, it is physically more interesting to find the maximum turning angle for a fixed upstream Mach number M_1. This calculation is somewhat more complicated because it necessarily involves the fluid equation of state. Since r is uniquely fixed by the normal upstream Mach number $M_{1n} \equiv M_1 \sin\beta$ we can write

$$\theta = \theta[\beta, r(M_{1n} = M_1 \sin\beta)]$$

Following the rules of differentiation then gives

$$(\partial\theta/\partial\beta)_{M1} = (\partial\theta/\partial\beta)_r + (\partial\theta/\partial r)_\beta (dr/dM_{1n}) M_1 \cos\beta$$

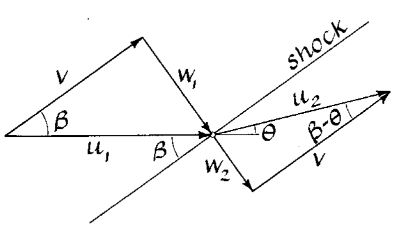

Fig. 1 Oblique shock geometry.

which is set equal to zero for an extremum. The partial derivatives on the right-hand side can be found from Eq. (2). Omitting the details, one then finds for the extremum

$$0 = r - \tan^2\beta + [M_{1n}/(r - 1)]dr/dM_{1n} \quad (5)$$

With $r = r(M_{1n} = M_1 \sin\beta)$, this is an implicit equation for the shock angle β_m' corresponding to maximum turning θ_m' at fixed M_1. In order to actually carry out the calculation, a particular relation $r(M_{1n})$ is needed: for example, a perfect gas has

$$r = (\gamma + 1)M_{1n}^2/[(\gamma - 1)M_{1n}^2 + 2]$$

It is still possible, however, to learn something about an arbitrary fluid. If the shock Hugoniot curve does not have a density minimum at finite pressure (see for example Hayes[3]), it follows that $dr/dM_{1n} \geq 0$ everywhere. Then Eq. (5) yields

$$\tan\beta_m' \geq r^{1/2} \quad (6)$$

where r is the density ratio at the extremum. With this result, Eq. (4) yields

$$\tan\theta_m' \leq (r - 1)/2r^{1/2} \quad (7)$$

If the shock Hugoniot does pass through a density minimum, the inequalities in Eqs. (6) and (7) would be reversed above that point.

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* Associate Professor, Division of Fluid, Chemical and Thermal Processes. Member AIAA.

Pressure and Velocity Fields in the Sondhauss Oscillator

John B. Miles* and Charles P. Mentesana†
University of Missouri, Columbia, Mo.

Introduction

This article considers the situation of heat addition to a gas-filled cavity which, under certain conditions, results in resonant acoustic oscillations. The general configuration studied in this investigation was originally conceived by Sondhauss,[1] but received very little additional attention until a dissertation by Feldman.[2] Reasons for this current interest include the similarity of the Sondhauss oscillations to the heat generated gas oscillations occurring in rocket engine combustion chambers and gas furnaces. Additionally, it has been speculatively proposed to incorporate the Sondhauss phenomenon into a plasma oscillator for use as an alternating-current MHD power generator,[3] or it could serve as a high intensity sound source for environmental testing.[4]

Prior to any practical exploitation of the Sondhauss oscillator, it is essential that a thorough understanding of the phenomenon be obtained. Accordingly, this article presents a summary of an experimental investigation[5] directed pri-

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* Professor, Department of Mechanical and Aerospace Engineering. Member AIAA.

† Former Graduate Student, Department of Mechanical and Aerospace Engineering.

marily at describing the pressure and velocity fields existing within the Sondhauss oscillator. Additionally, comparisons are made with the only other known experimental data.[2,6] In some instances these new results are either complementary or supporting of the previous data; in other cases they are in direct contradiction. As a result of inherent similarities, as well as to provide a convenient standard of comparison, the present data are also frequently compared to the one-quarter standing wave existing in a simple organ pipe arrangement. The similarity is shown to be quite close and complete.

Experimental System

The current experimental model (Fig. 1) was built according to the design criteria established by Feldman.[7] The 90-in. aluminum tube has one closed end and one open end, and results in a length to diameter ratio of 15. Holes in the ceramic core insulator permit air to pass between the heated and the unheated portions of the tube. Ten taps were placed in the tube surface along its length to allow for the insertion of pressure and velocity measuring probes.

Quarter Standing Wave

Information concerning the fluctuations of pressure and velocity existing in an organ pipe arrangement containing a quarter standing wave is readily available from acoustic theory. In particular, the pressure fluctuations at each axial location vary sinusoidally with time and are in phase with those at every other location. A similar situation exists for the velocity fluctuations, which furthermore lag the pressure fluctuations by 90° at every location (following the convention that velocities toward the open end of the pipe are positive).

Pressure fluctuation amplitudes are a maximum at the closed end and vary sinusoidally to zero at the open end. The opposite is true for velocity fluctuation amplitudes; i.e., maximum fluctuations exist at the open end. The resonant frequency (f) associated with a quarter standing wave is controlled by the sonic velocity (c_0) in the fluid and by the pipe length (L) through the relationship $f = c_0/4L$.

Pressure Measurements

Pressure fluctuation amplitudes, frequencies, and relative phase angles were measured at the ten tap locations for heater power settings of 610, 456, and 276 w. The primary measurements were taken with a quartz microphone, in conjunction with a second pressure transducer which served as a time base for phase determinations. As the primary transducer was successively moved from one tap location to another, both transducer outputs were viewed on an oscilloscope to provide the required information.

The resulting amplitude distributions are shown in Fig. 2, wherein the similarity to the quarter standing wave is obvious. Furthermore, the measured frequencies agreed quite closely with the equation $f = c_0/4L$, and the fluctuations at the various locations were noted to be in phase with one another; all in agreement with the quarter wave model. Additionally, the maximum pressure fluctuation amplitude was observed to vary as the square root of the heater power input, as might

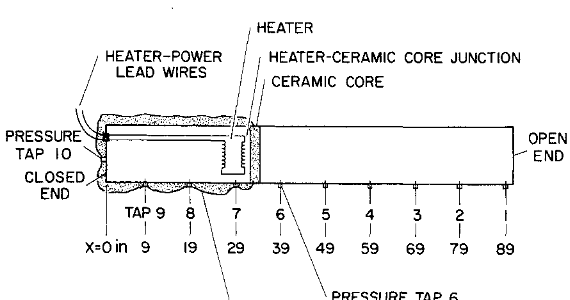

Fig. 1 The Sondhauss thermoacoustic oscillator.

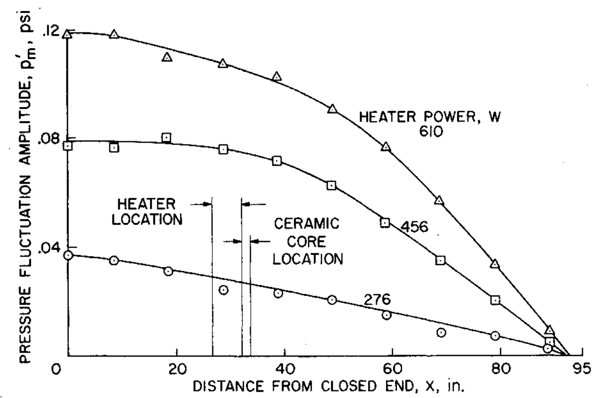

Fig. 2 Pressure fluctuation amplitude vs distance from the closed end of the tube for heater power inputs of 276, 456 and 610 ws.

be expected from the similar variation between acoustic power flow (sound intensity) and sound pressure fluctuation amplitude.

Velocity Measurements

Measurements of air velocity were made at all ten taps for power settings of 455 and 294 w. The measurements were accomplished by means of a constant temperature, linearized, hot wire anemometer system, capable of compensating for varying fluid temperature as exists near the heater elements. Results of these velocity fluctuation amplitude measurements are shown in Fig. 3. The indicated variations from zero near the closed end and increasing monotonously to a maximum at the open end are consistent with the quarter standing wave. Furthermore, the maximum velocity fluctuations (open end) were found to compare favorably with those predicted by acoustic theory for corresponding maximum pressure fluctuations as measured at the closed end.

The determination of phase relationships for the velocity fluctuations presented an interesting problem, inasmuch as a hot wire probe is inherently insensitive to the sense of the flow. This problem was overcome by placing two small windows of different diameters on opposite sides (axially) of the probe sensing wire. The resulting selective attenuation of flow approaching the wire in the positive direction as opposed to the negative direction allowed for the desired phase determinations. Simultaneous recordings at successive probe locations of a velocity fluctuation together with a single pressure signal (which were previously shown to be everywhere in phase) provided information on the phase differences between velocity and pressure fluctuations at each location. It was thereby determined that pressure leads velocity at every location by somewhat more than 90° (95° to 138°),

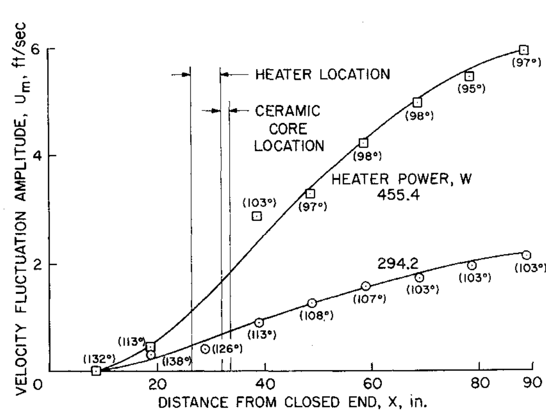

Fig. 3 Velocity fluctuation amplitude vs distance from the closed end of the tube for heater power inputs of 294 and 455 ws.

and these phase differences are indicated in parentheses on Fig. 3. The measured differences are reasonably consistent with the 90° figure existing for the quarter standing wave, and undoubtedly these deviations can be attributed to the flow obstruction caused by the tube bundle and to the severe temperature (and density) gradients occurring in the vicinity of the heater.

Discussion of Pressure Measurements

In a recent article summarizing their previous work, Feldman and Carter[9] present data of sound pressure vs axial position which is reasonably consistent with Fig. 2. However, they report a pressure maximum in the vicinity of the tube bundle, and they display their results in a decibel mode which makes comparison to the quarter standing wave or to the present results rather awkward. The frequency measurements reported by Feldman and Carter are likewise in agreement with the quarter wave prediction, but a theoretical model solved by them predicted frequencies almost twice those which were measured. They wrongly attributed[2] this discrepancy to experimenting with a double-length tube closed at both ends, wherein they actually had specified an erroneous boundary condition of zero pressure fluctuation rather than zero pressure gradient at a closed end.

Feldman and Carter[9] present an equation for pressure fluctuation amplitude for moderate heat inputs which indicates a linear variation with heat input. Their support for this variation is rather questionably based on sound pressure observations taken at the open end, at which point the pressure fluctuations are theoretically expected to be zero. Alternatively, for higher rates of heat release they contend that the pressure fluctuations at some unspecified point vary as the heat input to the $\frac{2}{3}$-power. Both of these variations are in contradiction to the present data and to the previously described expectations available from acoustic theory.

Finally, no previous data are available for indicating the relative phase for the pressure fluctuations at various axial locations.

Discussion of Velocity Measurements

In his original work, Feldman[2] made some qualitative velocity measurements with a homemade anemometer. He simultaneously viewed this velocity signal and a pressure signal to form Lissajous figures on an oscilloscope, and concluded that the local velocity fluctuation at all points leads the pressure fluctuation by approximately 20°. Inasmuch as Feldman considers velocities towards the closed end as positive, this would correspond to pressure leading velocity by 160° according to the present sign convention. However, it is difficult to place much confidence in these measurements since no provisions were made to render the anemometer probe sensitive to flow direction.

The only other comparable velocity data is that of Feldman and Hirsch,[7] who took some velocity measurements on a $3\frac{3}{8}$-in.-i.d. $\times$ 40-in.-long, open-ended Sondhauss tube. For the three heater power settings considered, they reported velocities of roughly the same magnitude as those herein reported, but their results strangely showed a decrease in velocity towards the open end, in sharp contrast to both the present data and the quarter wave model. A possible explanation for this discrepancy might be the fact that Feldman and Hirsch used a nonlinearized anemometer which was not temperature compensated, thereby leading to some complications in the regions of sharp temperature gradients created by the heater.

Feldman and Hirsch also measured phase differences between pressure and velocity fluctuations by means of an anemometer probe modified for sense determination in a similar manner to that previously described. No effort was made to determine relative phase for velocity fluctuations between one location and another, but rather velocity fluctuation was referenced to pressure fluctuation at each axial location. They reported results, in terms of the present sign convention, of pressure leading velocity at each location by 56° to 65°. This significant deviation from the present data and from the quarter wave model is especially suspect since this departure from the ideal value of 90° is reported for even the open end, where remoteness from the heating elements would logically bring the results more nearly into consistency with the quarter wave value. The only explanation herein offered is the same as cited in the previous paragraph.

Conclusions

It is contended that the most complete and consistent data currently available for describing the pressure and velocity fields existing within a Sondhauss oscillator have been herein presented. In view of the close approximation of these data to that predicted by the quarter wave model, it is suggested that any future investigations into the Sondhauss oscillator employ the quarter wave model as a standard of comparison.

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